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Factorizations of rational matrix functions with application to discrete isomonodromic transformations and difference Painlevé equations

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Abstract

We study factorizations of rational matrix functions with simple poles on the Riemann sphere. For the quadratic case (two poles) we show, using multiplicative representations of such matrix functions, that a good coordinate system on this space is given by a mix of residue eigenvectors of the matrix and its inverse. Our approach is motivated by the theory of discrete isomonodromic transformations and their relationship with difference Painlevé equations. In particular, in these coordinates, basic isomonodromic transformations take the form of the discrete Euler–Lagrange equations. Secondly we show that dPV equations, previously obtained in this context by D Arinkin and A Borodin, can be understood as simple relationships between the residues of such matrices and their inverses.

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1. Introduction

In this paper, we are concerned with various ways of introducing coordinates on the space of rational matrix functions $\mathbf{L}(z)$ (with simple poles) on the Riemann sphere. Examples like this play an important role in various applications, like Yang–Baxter maps and matrix solitons [1, 2], Lax equations and isomonodromy transformations on algebraic curves, [3, 4], discrete integrable systems, [5, 6] and others. Our main interest is related to the study of discrete isomonodromic transformations and their relationship with discrete Painlevé equations. The theory of discrete isomonodromic transformations was developed by A Borodin for polynomial matrices in [7] and adapted to the notion of local monodromy using rational matrix functions by Krichever in [8]. In [9], Arinkin and Borodin used the theory of d-connections on vector bundles to explain the appearance of difference Painlevé equations, considered from the geometric point of view of Sakai [10], in the theory of discrete isomonodromic transformations and later, in [11], they introduced the notion of the τ-function of such transformation. These τ-functions appear as the gap probabilities in the discrete probabilistic models of random matrix type.

Krichever conjectured that discrete isomonodromic transformations can be written in the Lagrangian form and that they should be related to the universal symplectic form of Krichever–Phong [12]. In [13], using the methods of [5], we verified this conjecture for the quadratic (two-pole) case by using the multiplicative coordinates on the space of these matrix functions and finding the explicit formula for the Lagrangian. Recently Soloviev showed that our Lagrangian symplectic form coincides with the reduction of the quadratic symplectic form of Krichever–Phong to certain symplectic leaves [14].

Unfortunately, the multiplicative coordinates are not easily obtained from other characteristic properties of $\mathbf{L}(z)$, such as its residue matrices. This is an obstacle to the generalization of our results to the higher order case. In this paper, we argue that one way around this difficulty is to consider, in addition to the residues of $\mathbf{L}(z)$, the residues of the inverse matrix $\mathbf{L}^{-1}(z)$. Then good coordinates on the space of $\mathbf{L}(z)$, again in the quadratic case, are given by half of the residue data of $\mathbf{L}(z)$ and half of the residue data of $\mathbf{L}^{-1}(z)$. Such residues also explain the almost symmetric form of the expressions for the multiplicative coordinates of $\mathbf{L}(z)$ and allow us to recognize dPV equations as simple relations between the residues of $\mathbf{L}(z)^{\pm 1}$ and the residues of the transformed matrices $\tilde{\mathbf{L}}(z)^{\pm 1}$.

The paper is organized as follows. In section 2 we give a short overview of the additive representation of $\mathbf{L}(z)$. In section 3 we study the multiplicative representation of $\mathbf{L}(z)$ and establish the relationship between the eigenvectors of the residues of $\mathbf{L}(z)^{\pm 1}$ and the eigenvectors of the left and right divisors of $\mathbf{L}(z)$. We also significantly simplify many of the arguments and formulas of [13]. Finally, in section 4 we restrict our attention to the rank-2 case. It this case, it is possible to introduce the so-called spectral coordinates on the space of $\mathbf{L}(z)$ in such a way that the equations relating the spectral coordinates of $\mathbf{L}(z)$ and $\tilde{\mathbf{L}}(z)$ are precisely the difference Painlevé equations. Our main observation here is the following. In addition to spectral coordinates, divisor of zeros and poles, and some asymptotic behavior at infinity, the entries of the matrix $\mathbf{L}(z)$ also depend on a choice of a gauge with respect to the action of the group of constant non-degenerate diagonal matrices. Understanding the change of this gauge from $\mathbf{L}(z)$ to $\mathbf{L}^{-1}(z)$ and $\tilde{\mathbf{L}}(z)$ allows us to easily obtain the expressions of the multiplicative representation of $\mathbf{L}(z)$ in the spectral coordinates and the dPV equations.

2. Additive form of rational matrix functions

Let $\mathbf{L}(z)$ be a rational matrix function on the Riemann sphere, rank $\mathbf{L}(z) = m$, satisfying the following general conditions. First, we require that there exists a normalization point z_0 at which $\mathbf{L}(z)$ is regular, $\lim_{z\to z_0}\mathbf{L}(z) = \mathbf{L}_0$, $\det \mathbf{L}_0 \neq 0$, and all eigenvalues of $\mathbf{L}_0$ are distinct. We can then use a gauge transformation to make $\mathbf{L}_0$ diagonal, thus reducing the global gauge group to the group of diagonal matrices. Without any loss of generality we can assume that $z_0 = \infty$, and so

$$\lim_{z\to\infty}\mathbf{L}(z) = \mathbf{L}_0 = \operatorname{diag}\{\rho_1,\ldots,\rho_m\}. \tag{2.1}$$

Second, we impose the following conditions on the pole structure of $\mathbf{L}(z)$ and its inverse $\mathbf{M}(z) = \mathbf{L}(z)^{-1}$. We require that $\mathbf{L}(z)$ is holomorphic except for *simple* poles at the points $z_1,\ldots,z_k$, $\mathbf{M}(z)$ is holomorphic except for *simple* poles at the points $\zeta_1,\ldots,\zeta_k$, all z_i and ζ_j are distinct, and the determinant $\det \mathbf{L}(z)$ has also only *simple* poles at z_i and *simple* zeros at ζ_j. These conditions mean that the residues $\mathbf{L}_i = \operatorname{res}_{z_i}\mathbf{L}(z)$ and $\mathbf{M}_j = -\operatorname{res}_{\zeta_j}\mathbf{M}(z)$ (the negative sign here is for future convenience) are matrices of rank 1. Using the † symbol to indicate a row vector, we have

$$\mathbf{L}(z)=\mathbf{L}_0+\sum_{i=1}^{k}\frac{\mathbf{L}_i}{z-z_i},\quad \text{where}\quad \mathbf{L}_0=\operatorname{diag}\{\rho_1,\ldots,\rho_m\}\quad\text{and}\quad \mathbf{L}_i=\mathbf{a}_i\mathbf{b}_i^{\dagger}, \tag{2.2}$$

$$\det\mathbf{L}(z)=\rho_1\ldots\rho_m\frac{\prod_{i=1}^{k}(z-\zeta_i)}{\prod_{j=1}^{k}(z-z_j)}, \tag{2.3}$$

$$\mathbf{L}(z)^{-1}=\mathbf{M}(z)=\mathbf{M}_0-\sum_{i=j}^{k}\frac{\mathbf{M}_j}{z-\zeta_j},\qquad \text{where}\quad \mathbf{M}_0=\mathbf{L}_0^{-1},\qquad \mathbf{M}_j=\mathbf{c}_j\mathbf{d}_j^{\dagger}. \tag{2.4}$$

We call the above representations of $\mathbf{L}(z)$ and $\mathbf{M}(z)$ *additive representations* and the vectors $\mathbf{a}_i,\mathbf{b}_i^{\dagger}$ (resp. $\mathbf{c}_i,\mathbf{d}_i^{\dagger}$) *additive eigenvectors* of $\mathbf{L}(z)$ (resp. $\mathbf{M}(z)$). Note that in terms of $\mathbf{L}(z)$ the vectors $\mathbf{c}_j$ (resp. $\mathbf{d}_j^{\dagger}$) can be characterized as the left (resp. right) null vectors of $\mathbf{L}(\zeta_j)$, and the similar statement is true for $\mathbf{M}(z_i)$ and $\mathbf{a}_i,\mathbf{b}_i^{\dagger}$. By the divisor of $\mathbf{L}(z)$ we mean the divisor $\mathcal{D}$ of its determinant, $\mathcal{D}=\sum_i z_i-\sum_i\zeta_i$. We denote the space of matrices $\mathbf{L}(z)$ satisfying conditions (2.2)–(2.4) by $\mathcal{M}_r^{\mathcal{D}}$.

An important question is how to choose a good coordinate system on the space $\mathcal{M}_r^{\mathcal{D}}$. For example, given $\mathbf{L}_i$, we can determine $\mathbf{a}_i$ and $\mathbf{b}_i^{\dagger}$ only up to a common scaling factor. This factor has to be adjusted to ensure that $\det\mathbf{L}(z)$ has zeros at ζ_i, which is a complicated condition on $\operatorname{tr}(\mathbf{L}_i)$. The same problem is present for the collection $\{\mathbf{c}_i,\mathbf{d}_i^{\dagger}\}$. Some insight for a good choice of coordinates is provided by the study of the isospectral and isomonodromic transformations, which suggests that half of the coordinates should be taken from the residues of $\mathbf{L}(z)$ and half from the residues of $\mathbf{M}(z)$. In fact, in the quadratic case, we have the following result.

Theorem 2.1. *When $\mathbf{L}(z)$ has $k=2$ poles, the vectors $(\mathbf{c}_2,\mathbf{d}_1^{\dagger};\mathbf{a}_2,\mathbf{b}_1^{\dagger})$, considered up to rescaling (i.e. as points in $\mathbb{P}^{r-1}$), are coordinates on the space $\mathcal{M}_r^{\mathcal{D}}$. To recover $\mathbf{L}^{\pm1}(z)$, consider the function*

$$\begin{aligned}\mathcal{L}\big((\mathbf{x}_2,\mathbf{x}_1^{\dagger}),(\mathbf{y}_2,\mathbf{y}_1^{\dagger})\big)&=(z_2-z_1)\log\big(\mathbf{x}_1^{\dagger}\mathbf{L}_0\mathbf{x}_2\big)+(z_1-\zeta_2)\log\big(\mathbf{y}_1^{\dagger}\mathbf{x}_2\big)\\&\quad+(\zeta_2-\zeta_1)\log\big(\mathbf{y}_1^{\dagger}\mathbf{L}_0^{-1}\mathbf{y}_2\big)+(\zeta_1-z_2)\log\big(\mathbf{x}_1^{\dagger}\mathbf{y}_2\big).\end{aligned} \tag{2.5}$$

Then

$$\mathbf{a}_1=-\frac{\partial\mathcal{L}}{\partial\mathbf{x}_1^{\dagger}}\big((\mathbf{c}_2,\mathbf{d}_1^{\dagger}),(\mathbf{a}_2,\mathbf{b}_1^{\dagger})\big);\qquad \mathbf{b}_2^{\dagger}=\frac{\partial\mathcal{L}}{\partial\mathbf{x}_2}\big((\mathbf{c}_2,\mathbf{d}_1^{\dagger}),(\mathbf{a}_2,\mathbf{b}_1^{\dagger})\big); \tag{2.6}$$

$$\mathbf{c}_1=\frac{\partial\mathcal{L}}{\partial\mathbf{y}_1^{\dagger}}\big((\mathbf{c}_2,\mathbf{d}_1^{\dagger}),(\mathbf{a}_2,\mathbf{b}_1^{\dagger})\big);\qquad \mathbf{d}_2^{\dagger}=-\frac{\partial\mathcal{L}}{\partial\mathbf{y}_2}\big((\mathbf{c}_2,\mathbf{d}_1^{\dagger}),(\mathbf{a}_2,\mathbf{b}_1^{\dagger})\big). \tag{2.7}$$

The proof of this theorem is based on the multiplicative representations of $\mathbf{L}(z)$, which we consider next.

3. Multiplicative form of rational matrix functions

3.1. Elementary divisors

To change from the additive to a multiplicative representation, we first need to define the building blocks for it. We call such building blocks the *elementary divisors*.

Definition 3.1. *An elementary divisor is a rational $m \times m$ matrix function $\mathbf{B}(z)$ on the Riemann sphere of the form*

$$\mathbf{B}(z) = \mathbf{I} + \frac{\mathbf{G}}{z - z_0}, \qquad \text{where} \quad \mathbf{G} = \mathbf{p}\mathbf{q}^{\dagger} \text{ is a matrix of rank 1.} \tag{3.1}$$

A direct calculation establishes the following elementary facts.

Lemma 3.2. *Let $B(z)$ be an elementary divisor. Then*

(i) $\det \mathbf{B}(z) = (z - \zeta_0)/(z - z_0)$, *where* $\zeta_0 = z_0 - \mathbf{q}^{\dagger}\mathbf{p}$;
(ii) $\mathbf{B}(z)^{-1} = \mathbf{I} - \mathbf{G}/(z - \zeta_0)$.

In fact, for us it will be more convenient to fix the points ζ_0 and z_0 on $\mathbb{CP}^1$. Thus, we say that a pair (ζ_0, z_0) corresponds to an elementary divisor $\mathbf{B}(z)$ with the determinant $\det \mathbf{B}(z) = (z - \zeta_0)/(z - z_0)$. Any generic matrix $\mathbf{B}(z)$ with such determinant and normalized by the condition $\mathbf{B}(z) \to \mathbf{I}$ as $z \to \infty$ is of the form $\mathbf{B}(z) = \mathbf{I} + \mathbf{G}/(z - z_0)$ with $\mathbf{G}$ of rank 1 and $\det \mathbf{B}(\zeta_0) = 0$ (more carefully, $\mathbf{B}(z)^{-1}$ should be regular at z_0 and $\mathbf{B}(z)$ should be regular at ζ_0). Thus, $\mathbf{B}(\zeta_0)$ has a left null vector $\mathbf{q}^{\dagger}$ and a right null vector $\mathbf{p}$. We then immediately get that $\mathbf{G} = \mathbf{p}\mathbf{q}^{\dagger}$, where we need to normalize the vectors $\mathbf{p}$ and $\mathbf{q}^{\dagger}$ so that $\mathbf{q}^{\dagger}\mathbf{p} = z_0 - \zeta_0$. In a more invariant form this can be written as

$$\mathbf{B}(z) = \mathbf{I} + \frac{z_0 - \zeta_0}{z - z_0} \frac{\mathbf{p}\mathbf{q}^{\dagger}}{\mathbf{q}^{\dagger}\mathbf{p}}. \tag{3.2}$$

From that point of view, the formula for the inverse matrix follows from the vanishing of the residue of the identity $\mathbf{B}(z)\mathbf{B}(z)^{-1} = \mathbf{I} = \mathbf{B}(z)^{-1}\mathbf{B}(z)$ at z_0,

$$\mathbf{B}(z)^{-1} = \mathbf{I} + \frac{\zeta_0 - z_0}{z - \zeta_0} \frac{\mathbf{p}\mathbf{q}^{\dagger}}{\mathbf{q}^{\dagger}\mathbf{p}}. \tag{3.3}$$

The following easy properties of elementary divisors are very useful for what follows.

Lemma 3.3. *Let*

$$\mathbf{B}(z) = \mathbf{I} + \frac{z_0 - \zeta_0}{z - z_0} \frac{\mathbf{p}\mathbf{q}^{\dagger}}{\mathbf{q}^{\dagger}\mathbf{p}}. \tag{3.4}$$

Then

(i)

$$\mathbf{B}(z)\mathbf{p} = \left(\frac{z - \zeta_0}{z - z_0}\right)\mathbf{p} \qquad \text{and} \qquad \mathbf{q}^{\dagger}\mathbf{B}(z) = \left(\frac{z - \zeta_0}{z - z_0}\right)\mathbf{q}^{\dagger}. \tag{3.5}$$

(ii) Suppose that at some point z^* we have $\mathbf{B}(z^*)\mathbf{w} = \mathbf{v}$ for some vectors $\mathbf{v}$ and $\mathbf{w}$. Then

$$\mathbf{B}(z) = \mathbf{I} + \frac{1}{z - z_0}\left((z_0 - z^*)\frac{\mathbf{w}\mathbf{q}^{\dagger}}{\mathbf{q}^{\dagger}\mathbf{w}} + (z^* - \zeta_0)\frac{\mathbf{v}\mathbf{q}^{\dagger}}{\mathbf{q}^{\dagger}\mathbf{v}}\right) \tag{3.6}$$

(i.e. we can determine $\mathbf{p}$ from $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{q}^{\dagger}$). Similarly, if $\mathbf{w}^{\dagger}\mathbf{B}(z^*) = \mathbf{v}^{\dagger}$,

$$\mathbf{B}(z) = \mathbf{I} + \frac{1}{z - z_0}\left((z_0 - z^*)\frac{\mathbf{p}\mathbf{w}^{\dagger}}{\mathbf{w}^{\dagger}\mathbf{p}} + (z^* - \zeta_0)\frac{\mathbf{p}\mathbf{v}^{\dagger}}{\mathbf{v}^{\dagger}\mathbf{p}}\right). \tag{3.7}$$

Proof. Part (i) is immediate. To establish the first formula in part (ii), we solve $\mathbf{B}(z^*)\mathbf{w} = \mathbf{v}$ for $\mathbf{p}/(\mathbf{q}^\dagger\mathbf{p})$ and then use (i):

$$\frac{\mathbf{p}}{\mathbf{q}^\dagger\mathbf{p}} = \frac{z_0 - z^*}{z_0 - \zeta_0}\frac{\mathbf{w}}{\mathbf{q}^\dagger\mathbf{w}} + \frac{z^* - z_0}{z_0 - \zeta_0}\frac{\mathbf{v}}{\mathbf{q}^\dagger\mathbf{w}} = \frac{z_0 - z^*}{z_0 - \zeta_0}\frac{\mathbf{w}}{\mathbf{q}^\dagger\mathbf{w}} + \frac{z^* - \zeta_0}{z_0 - \zeta_0}\frac{\mathbf{v}}{\mathbf{q}^\dagger\mathbf{v}}. \tag{3.8}$$

The second formula is proved in the similar way. □

Finally, we need the following notation.

Notation: We define a *twisting* of an elementary divisor $\mathbf{B}(z)$ by a constant non-degenerate matrix $\mathbf{A}$ to be a new elementary divisor ${}^{\mathbf{A}}\mathbf{B}(z)$ such that ${}^{\mathbf{A}}\mathbf{B}(z)\mathbf{A} = \mathbf{A}\mathbf{B}(z)$, i.e.

$${}^{\mathbf{A}}\mathbf{B}(z) = \mathbf{A}\mathbf{B}(z)\mathbf{A}^{-1} = \mathbf{I} + \frac{z_0 - \zeta_0}{z - z_0}\frac{(\mathbf{A}\mathbf{p})(\mathbf{q}^\dagger\mathbf{A}^{-1})}{(\mathbf{q}^\dagger\mathbf{A}^{-1})(\mathbf{A}\mathbf{p})}. \tag{3.9}$$

3.2. Factors and divisors

We begin with the following important remark. For the additive representation of $\mathbf{L}(z)$, the ordering of zeros and poles of $\det\mathbf{L}(z)$ is not important, but for any multiplicative representation choosing such an ordering is crucial. Thus, from now on our labeling will reflect the fact that the (ζ_s, z_s)-pair corresponds to some elementary divisor in $\mathbf{L}(z)$. There are two ways to look at the multiplicative structure of $\mathbf{L}(z)$—we can look at *factors* or at *divisors*.

Definition 3.4. *We say that elementary divisors $\mathbf{C}_s(z) = \mathbf{I} + \mathbf{H}_s/(z - z_s)$ corresponding to pairs (ζ_s, z_s) with $\mathbf{H}_s = \mathbf{m}_s\mathbf{n}_s^\dagger$ are the factors of $\mathbf{L}(z)$ if*

$$\mathbf{L}(z) = \mathbf{L}_0\mathbf{C}_1(z)\cdots\mathbf{C}_k(z). \tag{3.10}$$

Definition 3.5. *We say that elementary divisors $\mathbf{B}_s^r(z)$ (resp. $\mathbf{B}_s^l(z)$) corresponding to pairs (ζ_s, z_s) are right (resp. left) divisors of $\mathbf{L}(z)$ if $\mathbf{L}(z) = \mathbf{L}_s^r(z)\mathbf{B}_s^r(z)$ (resp. $\mathbf{L}(z) = \mathbf{B}_s^l(z)\mathbf{L}_s^l(z)$), where $\mathbf{L}_s^r(z)$ (resp. $\mathbf{L}_s^l(z)$) is regular at z_s.*

The main advantage of the divisors as opposed to the factors is that they can be written explicitly in terms of the residues of $\mathbf{L}(z)$ and $\mathbf{M}(z)$. Note that $\mathbf{C}_k(z) = \mathbf{B}_k^r(z)$ and ${}^{\mathbf{L}_0}\mathbf{C}_1(z) = \mathbf{B}_1^l(z)$. In particular, in the quadratic case $k = 2$, there is no essential difference between divisors and factors.

Lemma 3.6. *Let $\mathbf{L}_s = \operatorname{res}_{z_s}\mathbf{L}(z) = \mathbf{a}_s\mathbf{b}_s^\dagger$ and $\mathbf{M}_s = -\operatorname{res}_{\zeta_s}\mathbf{M}(z) = \mathbf{c}_s\mathbf{d}_s^\dagger$. Then*

$$\mathbf{B}_s^r(z) = \mathbf{I} + \frac{z_s - \zeta_s}{z - z_s}\frac{\mathbf{c}_s\mathbf{b}_s^\dagger}{\mathbf{b}_s^\dagger\mathbf{c}_s} \tag{3.11}$$

$$\mathbf{B}_s^l(z) = \mathbf{I} + \frac{z_s - \zeta_s}{z - z_s}\frac{\mathbf{a}_s\mathbf{d}_s^\dagger}{\mathbf{d}_s^\dagger\mathbf{a}_s}. \tag{3.12}$$

Proof. To obtain the formula for the right divisor, we take the residue of $\mathbf{L}(z) = \mathbf{L}_s^r(z)\mathbf{B}_s^r(z)$ at z_s to get $\mathbf{b}_s^\dagger \sim (\mathbf{q}_s^r)^\dagger$. Then we take the residue of $\mathbf{M}(z) = (\mathbf{B}_s^r(z))^{-1}(\mathbf{L}_s^r(z))^{-1}$ at ζ_s to get $\mathbf{c}_s \sim \mathbf{p}_s^r$. The formula then follows. The expression for the left divisors is obtained in the same way. □

3.3. Refactorization transformations

As shown in [7, 8], isomonodromic transformations on the space of rational matrix functions $\mathbf{L}(z)$ have the form

$$\mathbf{L}(z) \mapsto \tilde{\mathbf{L}}(z) = \mathbf{R}(z+1)\mathbf{L}(z)\mathbf{R}(z)^{-1}, \tag{3.13}$$

where $\mathbf{R}(z)$ is a certain rational matrix function. Similarly, we can consider isospectral transformations

$$\mathbf{L}(z) \mapsto \tilde{\mathbf{L}}(z) = \mathbf{R}(z)\mathbf{L}(z)\mathbf{R}(z)^{-1}. \tag{3.14}$$

In the isospectral case, we have to choose $\mathbf{R}(z)$ in such a way that the singularity structure of $\tilde{\mathbf{L}}(z)$ is the same as $\mathbf{L}(z)$, and in the isomonodromic case we require that $\mathbf{R}(z)$ induces integral shifts of certain parameters corresponding to the linear difference system given by $\mathbf{L}(z)$ (see [7, 9] for details). In particular, these requirements are satisfied if we take $\mathbf{R}(z)$ to be one of the divisors of $\mathbf{L}(z)$, which corresponds to changing the order of the factors in the multiplicative representation of $\mathbf{L}(z)$; in the isospectral case the divisor $\mathcal{D}$ is fixed and in the isomonodromic case $\mathcal{D}$ is shifted by an integer vector. In what follows we only consider transformations of this particular type. We focus on the isospectral case, since the isomonodromic case is very similar.

Let us now restrict our attention to the quadratic case and take $\mathbf{R}(z) = \mathbf{B}^r_1(z)$:

$$\mathbf{L}(z) = \mathbf{B}^l_2(z)\mathbf{L}_0\mathbf{B}^r_1(z) \mapsto \tilde{\mathbf{L}}(z) = \mathbf{B}^r_1(z)\mathbf{B}^l_2(z)\mathbf{L}_0 = \tilde{\mathbf{B}}^l_2(z)\mathbf{L}_0\tilde{\mathbf{B}}^r_1(z). \tag{3.15}$$

To study these transformations from the point of view of the discrete Euler–Lagrange equations, it is necessary to construct a configuration space $\mathcal{Q}$, a Lagrangian function $\mathcal{L} \in \mathcal{F}(\mathcal{Q}\times\mathcal{Q})$, and a map $\eta : \mathcal{Q}\times\mathcal{Q} \to \mathcal{M}^{\mathcal{D}}_r$ such that η maps the Lagrangian dynamics to the refactorization dynamics. That is, denoting by $\underset{\sim}{\mathbf{Q}}$ the previous and by $\tilde{\mathbf{Q}}$ the next point of the discrete dynamics, we want $\eta(\underset{\sim}{\mathbf{Q}}, \mathbf{Q}) = \mathbf{L}(z)$ and $\eta(\mathbf{Q}, \tilde{\mathbf{Q}}) = \tilde{\mathbf{L}}(z)$, where the shift map $\boldsymbol{\Phi} : (\underset{\sim}{\mathbf{Q}}, \mathbf{Q}) \to (\mathbf{Q}, \tilde{\mathbf{Q}})$ should satisfy the discrete Euler–Lagrange equations:

$$\frac{\partial\mathcal{L}}{\partial\mathbf{Y}}(\underset{\sim}{\mathbf{Q}}, \mathbf{Q}) + \frac{\partial\mathcal{L}}{\partial\mathbf{X}}(\mathbf{Q}, \tilde{\mathbf{Q}}) = 0, \tag{3.16}$$

see [5, 15] for the general description of this approach, and [13] for our specific situation. Writing

$$\tilde{\mathbf{L}}(z) = \mathbf{B}^r_1(z)\mathbf{B}^l_2(z)\mathbf{L}_0 = \tilde{\mathbf{B}}^l_1(z)\mathbf{L}_0\tilde{\mathbf{B}}^r_2(z) \tag{3.17}$$

and using the uniqueness of divisors, we see that $\mathbf{B}^r_1(z) = \tilde{\mathbf{B}}^l_1(z)$ and $\mathbf{B}^l_2(z)\mathbf{L}_0 = \mathbf{L}_0\tilde{\mathbf{B}}^r_2(z)$. From lemma 3.6 it then follows that $\mathbf{c}_1 = \tilde{\mathbf{a}}_1$, $\mathbf{b}^\dagger_1 = \tilde{\mathbf{d}}^\dagger_1$, $\mathbf{a}_2 = \mathbf{L}_0\tilde{\mathbf{c}}_2$ and $\mathbf{d}^\dagger_2\mathbf{L}_0 = \tilde{\mathbf{b}}^\dagger_2$. Since we want to parametrize $\mathbf{L}(z)$ by $(\underset{\sim}{\mathbf{Q}}, \mathbf{Q})$, we see that if we take $\mathbf{Q} = \left(\mathbf{a}_2, \mathbf{b}^\dagger_1\right)$ as first half of our coordinates, the second half should be $\underset{\sim}{\mathbf{Q}} = \left(\underset{\sim}{\mathbf{a}}_2, \underset{\sim}{\mathbf{b}}^\dagger_1\right) = \left(\mathbf{L}_0\mathbf{c}_2, \mathbf{d}^\dagger_1\right)$. Now the proof of theorem 2.1 is the same as the proof of theorem 3.1 of [13], and it is sketched below.

Proof. (theorem 2.1) Taking the residue of

$$\mathbf{L}(z) = \mathbf{B}^l_1(z)\mathbf{L}_0\mathbf{B}^r_2(z) = \mathbf{B}^l_2(z)\mathbf{L}_0\mathbf{B}^r_1(z) \tag{3.18}$$

at the point z_1 and comparing the row spaces of the resulting rank-1 matrices gives $\mathbf{d}^\dagger_1\mathbf{L}_0\mathbf{B}^r_2(z_1) = \mathbf{b}^\dagger_1$. Using lemma 3.3 (ii) we can find the formula for $\mathbf{b}^\dagger_2$,

$$\mathbf{b}^\dagger_2 = (z_2 - z_1)\frac{\mathbf{d}^\dagger_1\mathbf{L}_0}{\mathbf{d}^\dagger_1\mathbf{L}_0\mathbf{c}_2} + (z_1 - \zeta_2)\frac{\mathbf{b}^\dagger_1}{\mathbf{b}^\dagger_1\mathbf{c}_2} = \frac{\partial\mathcal{L}}{\partial\mathbf{x}_2}\left(\left(\mathbf{c}_2, \mathbf{d}^\dagger_1\right), \left(\mathbf{a}_2, \mathbf{b}^\dagger_1\right)\right), \tag{3.19}$$

where $\mathcal{L}\left(\left(\mathbf{x}_2, \mathbf{x}^\dagger_1\right), \left(\mathbf{y}_2, \mathbf{y}^\dagger_1\right)\right)$ is given by (2.5). Performing similar calculations at the point z_2 for $\mathbf{L}(z)$ and the points ζ_1, ζ_2 for $\mathbf{M}(z)$ gives the rest of formulas (2.6)–(2.7) and completes the proof. □

In these coordinates theorem 3.1 of [13] takes the following form (here we improve the formulas from [13] by using $\mathbf{L}_0$ instead of its root).

Theorem 3.7.

(i) The map $\eta : \mathcal{Q}\times\mathcal{Q}\to\mathcal{M}_r^{\mathcal{D}}$ is given by

$$\eta(\underset{\sim}{\mathbf{Q}},\mathbf{Q}) = \mathbf{L}(z) = \left(\mathbf{I} + \frac{1}{z-z_2}\left((z_2-\zeta_1)\frac{\mathbf{a}_2\underset{\sim}{\mathbf{b}}_1^\dagger}{\underset{\sim}{\mathbf{b}}_1^\dagger\mathbf{a}_2} + (\zeta_1-\zeta_2)\frac{\mathbf{a}_2\mathbf{b}_1^\dagger\mathbf{L}_0^{-1}}{\mathbf{b}_1^\dagger\mathbf{L}_0^{-1}\mathbf{a}_2}\right)\right)\mathbf{L}_0 \times\left(\mathbf{I}+\frac{1}{z-z_1}\left((z_1-\zeta_2)\frac{\mathbf{L}_0^{-1}\underset{\sim}{\mathbf{a}}_2\mathbf{b}_1^\dagger}{\mathbf{b}_1^\dagger\mathbf{L}_0^{-1}\underset{\sim}{\mathbf{a}}_2} + (\zeta_2-\zeta_1)\frac{\mathbf{L}_0^{-1}\mathbf{a}_2\mathbf{b}_2^\dagger}{\mathbf{b}_2^\dagger\mathbf{L}_0^{-1}\mathbf{a}_2}\right)\right). \tag{3.20}$$

(ii) The equations of motion $(\mathbf{Q},\tilde{\mathbf{Q}}) = \boldsymbol{\Phi}(\underset{\sim}{\mathbf{Q}},\mathbf{Q})$ of both the isospectral and isomonodromic dynamics in these coordinates are given by the discrete Euler–Lagrange equations with the Lagrangian function

$$\mathcal{L}(\mathbf{X},\mathbf{Y},t) = (z_2 - z_1(t))\log\left(\mathbf{x}_1^\dagger\mathbf{x}_2\right) + (z_1(t)-\zeta_2)\log\left(\mathbf{y}_1^\dagger\mathbf{L}_0^{-1}\mathbf{x}_2\right) + (\zeta_2-\zeta_1(t))\log\left(\mathbf{y}_1^\dagger\mathbf{L}_0^{-1}\mathbf{y}_2\right) + (\zeta_1(t)-z_2)\log\left(\mathbf{x}_1^\dagger\mathbf{y}_2\right), \tag{3.21}$$

where in the isomonodromic case $z_1(t) = z_1 - t$, $\zeta_1(t) = \zeta_1 - t$, and in the isospectral case $z_1(t) = z_1$, $\zeta_1(t) = \zeta_1$ and $\mathcal{L}(\mathbf{X},\mathbf{Y})$ is time independent.

4. Isomonodromic transformations and dPV

In [9], Arinkin and Borodin showed that for rank-2 matrices isomonodromic transformations above, when written in a special coordinate system, are given by the difference Painlevé equations. In this section, choosing the dPV case as an example, we show that these equations appear explicitly as relations between the residues of $\mathbf{L}^{\pm1}(z)$ and $\tilde{\mathbf{L}}^{\pm1}(z)$. Similar computation for the q-PVI case was done earlier by Jimbo and Sakai [16].

4.1. Spectral coordinates

In the quadratic (two-pole) case the space of the rank-2 matrices $\mathbf{L}(z)$ satisfying the requirements (2.2)–(2.3) can be described using different parameters. These parameters come in two groups. The first group, that we call the *type* of $\mathbf{L}(z)$, consists of the zeros and poles of the determinant of $\mathbf{L}(z)$ and some asymptotic data at $z=\infty$. The space of $\mathbf{L}(z)$ of the fixed type is two dimensional, and the second group of parameters is a special coordinate system on this space, called the *spectral coordinates*. Expressing isomonodromy transformation in those coordinates gives rise to the difference Painlevé equations.

Definition 4.1. *Let $\mathbf{L}(z)$ be a rational 2×2 matrix function on the Riemann sphere satisfying the following conditions:*

$$\mathbf{L}(z) = \operatorname{diag}\{\rho_1,\rho_2\} + \frac{\mathbf{L}_1}{z-z_1} + \frac{\mathbf{L}_2}{z-z_2}, \qquad \mathbf{L}_i = \mathbf{a}_i\mathbf{b}_i^\dagger, \qquad \det\mathbf{L}(z) = \rho_1\rho_2\frac{(z-\zeta_1)(z-\zeta_2)}{(z-z_1)(z-z_2)}. \tag{4.1}$$

In addition, put

$$\rho_1 k_1 = (\mathbf{L}_\infty)_{11}, \qquad \rho_2 k_2 = (\mathbf{L}_\infty)_{22}, \qquad \mu = (\mathbf{L}_\infty)_{21}, \qquad \textit{where}\quad \mathbf{L}_\infty = -\operatorname{res}_\infty \mathbf{L}(z)\,\mathrm{d}z = \mathbf{L}_1+\mathbf{L}_2. \tag{4.2}$$

We call $(\rho_1, \rho_2, \zeta_1, \zeta_2, z_1, z_2, k_1, k_2, \mu)$ *the type of* $\mathbf{L}(z)$. *These parameters are not independent, since*

$$k_1 + k_2 = \operatorname{tr} \mathbf{L}_0^{-1}\mathbf{L}_\infty = (z_1 - \zeta_1) + (z_2 - \zeta_2). \tag{4.3}$$

The conditions on $\rho_i k_i$ *correspond to fixing the formal type of the solution of the difference equation at infinity, and the choice of* μ *corresponds to fixing the gauge under the global action by constant non-degenerate diagonal matrices.*

Definition 4.2. *The spectral coordinates* (γ, π) *are defined by the conditions that*

- $\mathbf{L}(\gamma)_{21} = 0$ *(and therefore* $\mathbf{L}(z)_{21} = \frac{\mu(z-\gamma)}{(z-z_1)(z-z_2)}$*);*
- $\pi = \frac{(\gamma-z_1)}{(\gamma-\zeta_2)}\mathbf{L}(\gamma)_{11}$.

The normalization conditions in this definition are chosen to match the formulas in [9] and [11].

Notation: To find the expression of $\mathbf{L}(z)$ and $\mathbf{M}(z)$ in the spectral coordinates, it is convenient to introduce the notation $\varphi(a, b) = \pi(\gamma - a) - \rho_1(\gamma - b)$.

Lemma 4.3. *Let* $\mathbf{L}(z)$ *be a rational* 2×2 *matrix function on the Riemann sphere that has the type* $(\rho_1, \rho_2, \zeta_1, \zeta_2, z_1, z_2, k_1, k_2, \mu)$. *Then, in spectral coordinates, the residues of the matrix* $\mathbf{L}(z)$ *are given by*

$$\mathbf{L}_1 = \mu\frac{\gamma - z_1}{z_2 - z_1}\begin{bmatrix} \frac{1}{\mu}\left(\rho_1 k_1 - \frac{(\gamma-z_2)}{\gamma-z_1}\varphi(\zeta_2, z_1)\right) \\ 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{\mu}\left(\rho_2 k_2 + \frac{\rho_2}{\pi}\varphi(z_2, \zeta_1)\right) \end{bmatrix}, \tag{4.4}$$

$$\mathbf{L}_2 = \mu\frac{\gamma - z_2}{z_1 - z_2}\begin{bmatrix} \frac{1}{\mu}\left(\rho_1 k_1 - \varphi(\zeta_2, z_1)\right) \\ 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{\mu}\left(\rho_2 k_2 + \frac{\rho_2(\gamma - z_1)}{\pi(\gamma - z_2)}\varphi(z_2, \zeta_1)\right) \end{bmatrix}. \tag{4.5}$$

Proof. Let $\mathbf{L}_i = \alpha_i \begin{bmatrix} a_i \\ 1 \end{bmatrix}\begin{bmatrix} 1 & b_i \end{bmatrix}$. Then

$$\mathbf{L}(z)_{21} = \frac{\alpha_1}{z - z_1} + \frac{\alpha_2}{z - z_2} = \frac{\mu(z-\gamma)}{(z-z_1)(z-z_2)}, \tag{4.6}$$

and so $\alpha_1 = \mu(\gamma - z_1)/(z_2 - z_1)$, $\alpha_2 = \mu(\gamma - z_2)(z_1 - z_2)$ and $\alpha_1 + \alpha_2 = \mu$.

The normalization at infinity and the definition of π,

$$\rho_1 k_1 = \alpha_1 a_1 + \alpha_2 a_2 = \mu a_1 + \alpha_2(a_2 - a_1) = \alpha_1(a_1 - a_2) + \mu a_2, \tag{4.7}$$

$$\mathbf{L}(\gamma)_{11} = \pi\frac{(\gamma - \zeta_2)}{(\gamma - z_1)} = \rho_1 + \frac{\alpha_1(a_1 - a_2)}{(\gamma - z_1)} = \rho_1 + \frac{\alpha_2(a_2 - a_1)}{(\gamma - z_2)}, \tag{4.8}$$

immediately give

$$a_1 = \frac{1}{\mu}(\rho_1 k_1 - \alpha_2(a_2 - a_1)) = \frac{1}{\mu}\left(\rho_1 k_1 - \frac{(\gamma - z_2)}{(\gamma - z_1)}\varphi(\zeta_2, z_1)\right), \tag{4.9}$$

$$a_2 = \frac{1}{\mu}\left(\rho_1 k_1 - \varphi(\zeta_2, z_1)\right). \tag{4.10}$$

Using the equation $\mathbf{L}(\gamma)_{11}\mathbf{L}(\gamma)_{22} = \det \mathbf{L}(\gamma)$ we get $\mathbf{L}(\gamma)_{22} = \frac{\rho_1\rho_2}{\pi}\frac{(\gamma-\zeta_1)}{(\gamma-z_2)}$. This, and the condition $\rho_2 k_2 = \alpha_1 b_1 + \alpha_2 b_2$, allows us to find the expressions for b_1, b_2 in exactly the same way. □

Corollary 4.4. *In the same gauge, the residues* $\mathbf{M}_i$ *of the inverse matrix*

$$\mathbf{M}(z) = \mathbf{L}(z)^{-1} = \operatorname{diag}\{1/\rho_1, 1/\rho_2\} - \frac{\mathbf{M}_1}{z-\zeta_1} - \frac{\mathbf{M}_2}{z-\zeta_2} \tag{4.11}$$

are given by

$$\mathbf{M}_1 = \frac{\mu}{\rho_1\rho_2}\frac{\gamma-\zeta_1}{\zeta_2-\zeta_1}\begin{bmatrix}\frac{1}{\mu}\left(\rho_2 k_1 - \frac{\rho_2}{\pi}\varphi(\zeta_2, z_1)\right)\\ 1\end{bmatrix}\begin{bmatrix}1 & \frac{1}{\mu}\left(\rho_1 k_2 + \frac{\gamma-\zeta_2}{\gamma-\zeta_1}\varphi(z_2,\zeta_1)\right)\end{bmatrix}, \tag{4.12}$$

$$\mathbf{M}_2 = \frac{\mu}{\rho_1\rho_2}\frac{\gamma-\zeta_2}{\zeta_1-\zeta_2}\begin{bmatrix}\frac{1}{\mu}\left(\rho_2 k_1 - \frac{\rho_2(\gamma-\zeta_1)}{\pi(\gamma-\zeta_2)}\varphi(\zeta_2, z_1)\right)\\ 1\end{bmatrix}\begin{bmatrix}1 & \frac{1}{\mu}\left(\rho_1 k_2 + \varphi(z_2,\zeta_1)\right)\end{bmatrix}. \tag{4.13}$$

Proof. Since $\mathbf{M}(z)$ has the same form as $\mathbf{L}(z)$, we only have to determine the type and spectral coordinates of $\mathbf{M}(z)$ in terms of those of $\mathbf{L}(z)$,

$\mathbf{L}(z)$:	z_1	z_2	ζ_1	ζ_2	ρ_1	ρ_2	k_1	k_2	μ	γ	π
$\mathbf{M}(z)$:	ζ_1	ζ_2	z_1	z_2	$\frac{1}{\rho_1}$	$\frac{1}{\rho_2}$	$-k_1$	$-k_2$	$-\frac{\mu}{\rho_1\rho_2}$	γ	$\frac{(\gamma-z_1)(\gamma-\zeta_1)}{\pi(\gamma-z_2)(\gamma-\zeta_2)}$

and take the negative sign in the definitions of $\mathbf{M}_i$ into account. Note that in computing the type of $\mathbf{M}(z)$ we used the equation $\mathbf{M}_\infty = -\mathbf{L}_0^{-1}\mathbf{L}_\infty\mathbf{L}_0^{-1}$, which follows from the condition $\operatorname{res}_\infty \mathbf{L}(z)\mathbf{M}(z) = \mathbf{0}$. □

4.2. Difference Painlevé V

Consider now the isomonodromy transformation given by $\mathbf{R}(z) = \mathbf{B}_1^r(z)$:

$$\mathbf{L}(z) = \mathbf{B}_2^l(z)\mathbf{L}_0\mathbf{B}_1^r(z) \mapsto \tilde{\mathbf{L}}(z) = \mathbf{B}_1^r(z+1)\mathbf{B}_2^l(z)\mathbf{L}_0 = \tilde{\mathbf{B}}_1^l(z)\mathbf{L}_0\tilde{\mathbf{B}}_2^r(z). \tag{4.14}$$

Theorem 4.5. *The type and spectral coordinates of* $\tilde{\mathbf{L}}(z)$ *in terms of those of* $\mathbf{L}(z)$ *are given by*

$\mathbf{L}(z)$:	z_1	z_2	ζ_1	ζ_2	ρ_1	ρ_2	k_1	k_2	μ	γ	π
$\tilde{\mathbf{L}}(z)$:	$\tilde{z}_1 = z_1 - 1$	$\tilde{z}_2 = z_2$	$\tilde{\zeta}_1 = \zeta_1 - 1$	$\tilde{\zeta}_2 = \zeta_2$	ρ_1	ρ_2	k_1	k_2	$\tilde{\mu}$	$\tilde{\gamma}$	$\tilde{\pi}$,

(4.15)

where

$$\tilde{\mu} = \mu\frac{\rho_1(\pi-\rho_2)}{\rho_2(\pi-\rho_1)} \tag{4.16}$$

$$\tilde{\gamma} + \gamma = z_2 + \zeta_2 + \frac{\rho_1(k_1 - z_1 + \zeta_2)}{\pi - \rho_1} + \frac{\rho_2(k_2 - z_1 + \zeta_2 + 1)}{\pi - \rho_2} \tag{4.17}$$

$$\tilde{\pi}\pi = \rho_1\rho_2\frac{(\tilde{\gamma}-\tilde{z}_1)(\tilde{\gamma}-\tilde{\zeta}_1)}{(\tilde{\gamma}-\tilde{z}_2)(\tilde{\gamma}-\tilde{\zeta}_2)}. \tag{4.18}$$

Equations (4.17)–(4.18) are the difference Painlevé V equations of Sakai's hierarchy [10], first obtained in this setting in [9] (Theorem B).

Proof. First note that

$$\mathbf{L}_\infty = \mathbf{G}_2^l\mathbf{L}_0 + \mathbf{L}_0\mathbf{G}_1^r, \qquad \tilde{\mathbf{L}}_\infty = \left(\mathbf{G}_1^r + \mathbf{G}_2^l\right)\mathbf{L}_0. \tag{4.19}$$

Thus, using (3.11), (4.4) and (4.12), we get

$$\tilde{\mu} = (\tilde{\mathbf{L}}_\infty)_{21} = \mu + \left[\mathbf{G}_1^r, \mathbf{L}_0\right]_{21} = \mu + (\rho_1 - \rho_2)\left(\mathbf{G}_1^r\right)_{21} = \mu + (\rho_1 - \rho_2)\frac{z_1 - \zeta_1}{\mathbf{b}_1^\dagger \mathbf{c}_1}$$
$$= \mu\left(1 + \frac{\pi(\rho_1 - \rho_2)}{\rho_2(\pi - \rho_1)}\right) = \mu\frac{\rho_1(\pi - \rho_2)}{\rho_2(\pi - \rho_1)}.$$

From the uniqueness of the left and right divisors, we see that $\mathbf{B}_1^r(z+1) = \tilde{\mathbf{B}}_1^l(z)$ and $\mathbf{B}_2^l(z) = {}^{\mathbf{L}_0}\tilde{\mathbf{B}}_2^r(z)$. Using the first equation, we see that $\mathbf{G}_1^r = \tilde{\mathbf{G}}_1^l$ and so $\left(\mathbf{c}_1\mathbf{b}_1^\dagger\right)/\left(\mathbf{b}_1^\dagger\mathbf{c}_1\right) = \left(\tilde{\mathbf{a}}_1\tilde{\mathbf{d}}_1^\dagger\right)/\left(\tilde{\mathbf{d}}_1^\dagger\tilde{\mathbf{a}}_1\right)$. In particular,

$$\frac{\tilde{\mu}}{\mu}\mathbf{b}_1^\dagger\mathbf{c}_1 = (z_1 - \zeta_1)\frac{\rho_1(\pi - \rho_2)}{\pi} = \tilde{\mathbf{d}}_1^\dagger\tilde{\mathbf{a}}_1 = (\tilde{z}_1 - \tilde{\zeta}_1)\left(\rho_1 - \frac{\tilde{\pi}(\tilde{\gamma} - \tilde{z}_2)(\tilde{\gamma} - \tilde{\zeta}_2)}{(\tilde{\gamma} - \tilde{z}_1)(\tilde{\gamma} - \tilde{\zeta}_1)}\right), \tag{4.20}$$

which gives (4.18). Similarly, comparing the first components of the normalized vectors $\mathbf{c}_1$ and $\tilde{\mathbf{a}}_1$ gives (4.17) and completes the proof. □

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